

A Barsotti–Weil isomorphism for Taelman units

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ABSTRACT

The positive-characteristic analogue of motivic cohomology with coefficients in an Anderson A-motive $\underline{M}$ over a function field of transcendence degree one was introduced in [Gaz22]. In this paper, we prove that, when $\underline{M}$ arises as the dual of the A-motive of an Anderson module E that is abelian, uniformizable, and has everywhere good reduction, the corresponding A-motivic cohomology coincides with the module of Taelman units of E, up to a twist by the module of Kähler differentials of A. To this end, we recast the theory of Anderson generating functions in the spirit of earlier work of Maurischat and the author on special functions.

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1 Introduction

Let A be an abelian variety over the field of rational numbers $\mathbb{Q}$. The Barsotti–Weil formula implies a canonical isomorphism between the group of $\mathbb{Q}$ -valued points of A and the extension group of commutative fppf sheaves on the big fppf site over $\mathbb{Q}$ between the dual abelian variety of $\mathcal{A}$ and $\mathbb{G}_m$:

$$\mathcal{A}(\mathbb{Q}) \cong \mathrm{Ext}_{\mathrm{AbGrpSch}/\mathbb{Q}}^1(\mathcal{A}^\vee, \mathbb{G}_m);$$

e.g. [BVK16, Lemma 1.13.3]. Its motivic counterpart is the expected canonical isomorphism

$$\mathcal{A}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{Ext}_{\mathrm{MxdMot}_{\mathbb{Q}}}^1(\mathbb{1}, h^1(\mathcal{A})^\vee) \quad (1.1)$$

where the right-hand side stands for the extension group in a hypothetical $\mathbb{Q}$ -linear category of mixed motives. There is, for instance, an unconditional 1-motive incarnation of the above

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(see [Del74, §10] and [BVK16, Theorem 2.1.2 and Appendix C, Proposition C.8.3(b)]). By the Mordell–Weil theorem, the left hand side of (1.1) is finite dimensional $\mathbb{Q}$ -vector space, which makes this interpretation particularly useful in arithmetic applications.

The aim of this paper is to prove two Barsotti–Weil-style isomorphisms in function field arithmetic, where abelian varieties are replaced by Anderson modules. Let $\mathbb{F}$ be a finite field with q elements and let $(C, \mathcal{O}_C)$ be a geometrically irreducible smooth projective curve over $\mathbb{F}$ marked with a closed point ∞ . Consider the following rings of algebraic functions on C :

$$A := \mathcal{O}_C(C \setminus \{\infty\}) \quad \text{and} \quad K := \text{Frac}(A) = \mathbb{F}(C).$$

The counterpart of abelian varieties but with coefficients in A was introduced by Anderson as a higher-dimensional generalization of Drinfeld modules [And86]. An Anderson module E over an A -algebra R is a particular instance of a smooth A -module scheme over R (see Subsection 2.1 for a complete definition). Anderson developed their theory of motives, called *A-motives*, which consist of certain $A \otimes_{\mathbb{F}} R$ -modules M equipped with a semilinear endomorphism of M with respect to the q -Frobenius on R . If E is an Anderson module over an A -algebra R , its associated A -motive is

$$\underline{M}(E) := \text{Hom}_{\mathbb{F}}(E, \mathbb{G}_a),$$

where homomorphisms are those of $\mathbb{F}$ -vector space schemes over R . The functor $\underline{M}$ shares many similarities with the functor h^1 in the number field setting.

We establish in this paper a first Barsotti–Weil-type isomorphism (see Theorem 2.7).

Theorem. *Let E be an Anderson module over an A -algebra R and let $\underline{M}(E)$ denote its A -motive. Then we have*

$$E(R) \cong \text{Ext}_{A \otimes_{\mathbb{F}} R[\tau]}^1(\underline{M}(E), A \otimes R) \otimes_A \Omega_{A/\mathbb{F}}^1$$

where the extension module is taken in the category of $A \otimes_{\mathbb{F}} R[\tau]$ -modules. When E is abelian, it implies

$$E(R) \cong \text{Ext}_{A \text{Mot}_R}^{1, \text{reg}}(\mathbb{1}, \underline{M}(E)^{\vee}) \otimes_A \Omega_{A/\mathbb{F}}^1 \quad (1.2)$$

where the extension module is now taken in the category of A -motives over R ; the superscript “reg” means regulated extensions, a notion defined in [Gaz24] to allow the construction of a regulator map.

This theorem was known in several cases. If E is abelian, $A = \mathbb{F}[t]$ and R is an algebraically closed field, it was already established by Taelman [Tae20]. For Drinfeld modules, by Mornev [Mor18] (in the language of shtukas), and for abelian Anderson modules over perfect R by the author and Maurischat [GM25].

Let F be a finite extension of K and let $\mathcal{O}_F$ be the integral closure of A in F . In contrast with the number-field situation, the finite generation of $E(\mathcal{O}_F)$ fails already in dimension one. The correct replacement is the submodule of $E(\mathcal{O}_F)$ consisting in *Taelman units* and is denoted by $U(E/\mathcal{O}_F)$ (see [Tae10] for Drinfeld modules). A natural problem is then to identify the extension submodule to which it corresponds to through the isomorphism (1.2). The aim of this paper is to show that $U(E/\mathcal{O}_F)$ agrees with the submodule of extensions having analytic reduction at ∞ , introduced in [Gaz22].

In order to formulate our statement precisely, let us introduce additional notation. We write (K_{∞}, v_{∞}) for the complete discrete valuation field of C at ∞ and let F_{∞} be $F \otimes_K K_{\infty}$. After fixing a separable closure K_{∞}^s of K_{∞} , we also write F_{∞}^s for $F \otimes_K K_{\infty}^s$. Observe that since K_{∞} is a separable field extension of K , both F_{∞} and F_{∞}^s are finite products of fields.

We now assume that E is a uniformizable Anderson A -module over $\mathcal{O}_F$; by definition, this means that its exponential map $\exp_E : \text{Lie}_E(F_\infty^s) \rightarrow E(F_\infty^s)$ is surjective and equivariant with respect to the profinite group $G_\infty := \text{Gal}(K_\infty^s|K_\infty)$. Taking invariants under its action yields a surjective connecting homomorphism to continuous Galois cohomology

$$E(F_\infty) \longrightarrow H^1(G_\infty, \Lambda_E) \quad (1.3)$$

where $\Lambda_E := \ker(\exp_E)$ is the period lattice of E , see [Tae10, §2]. We define Taelman's map t_{exp} as the restriction of (1.3) to $E(\mathcal{O}_F)$, and set

$$U(E/\mathcal{O}_F) := \ker(t_{\text{exp}}) \quad \text{and} \quad H(E/\mathcal{O}_F) := \text{coker}(t_{\text{exp}}),$$

respectively for the *Taelman unit module* and the *Taelman class module* of E . Taelman's unit theorem [Tae10, Theorem 1] can be stated as follows.

Theorem 1.1 (Taelman). *As A -modules, $U(E/\mathcal{O}_F)$ is finitely generated and $H(E/\mathcal{O}_F)$ is finite.*

The second Barsotti–Weil-type isomorphism connects Taelman's modules to the motivic invariants introduced in [Gaz22]. For this, let us further assume that E is abelian over $\mathcal{O}_F$; in particular, E_F has everywhere good reduction. By definition, this means that $\underline{M}(E)$ is a finite projective $A \otimes \mathcal{O}_F$ -module. Because E is also assumed to be uniformizable, $\underline{M}(E)_F$ is rigid analytically trivial (cf. [And86, BH07]). To lighten notations, we set $\underline{M} := \underline{M}(E)_F^\vee = \text{Hom}_{A \otimes \mathcal{O}_F}(\underline{M}(E)_F, \mathbb{1})$ which is also rigid analytically trivial with good reduction. The “Betti realization” functor from the category uniformizable A -motives to that of continuous G_∞ -representations is exact, and therefore induces a map

$$r_B : \text{Ext}_{\mathcal{O}_F}^{1, \text{reg}}(\mathbb{1}, \underline{M}) \longrightarrow H^1(G_\infty, \underline{M}_B)$$

when restricted to *regulated and integral extensions* (see [Gaz24] for definitions and [Gaz22, (3.6)] for details on the map r_B). Our main theorem is the following.

Theorem 1.2. *There is a natural commutative square of A -modules*

$$\begin{array}{ccc} \text{Ext}_{\mathcal{O}_F}^{1, \text{reg}}(\mathbb{1}, \underline{M}) \otimes_A \Omega_{A/\mathbb{F}}^1 & \xrightarrow{r_B} & H^1(G_\infty, \underline{M}_B) \otimes_A \Omega_{A/\mathbb{F}}^1 \\ \downarrow \wr & & \downarrow \wr \\ E(\mathcal{O}_F) & \xrightarrow{t_{\text{exp}}} & H^1(G_\infty, \Lambda_E) \end{array}$$

whose vertical arrows are isomorphisms, as indicated.

The theorem implies in particular that the module $\text{Ext}_{\mathcal{O}_F, \infty}^{1, \text{reg}}(\mathbb{1}, \underline{M}) \otimes_A \Omega_{A/\mathbb{F}}^1$ —defined as the kernel of r_B —is naturally isomorphic to Taelman's unit module $U(E/\mathcal{O}_F)$ and that $\text{Cl}(\underline{M}) \otimes_A \Omega_{A/\mathbb{F}}^1$ —defined as the cokernel of r_B —is isomorphic to $H(E/\mathcal{O}_F)$. In particular, [Gaz22, Theorem 6.1] recovers Taelman's theorem.

This shows that the formalism of A -motivic cohomology provides a reasonable definition of Taelman units/class modules for A -motives that do not arise from Anderson A -modules, or for abelian A -modules not having everywhere good reduction. An important benefit is that A -motivic cohomology might be easier to compute than Taelman units: in the case of Carlitz tensor powers where $A = \mathbb{F}[t]$, the former has been partly computed in [GM22], and the following corollary can be deduced.

Corollary 1.3. *Let $C^{\otimes n}$ denote the n -th tensor power of the Carlitz module over $\mathcal{O}_F$, where n is large compared to the genus of F . For $\alpha \in \mathcal{O}_F$, denote by $[\alpha]_n := (0, \dots, 0, -\alpha)$ the symbol of α in $C^{\otimes n}(\mathcal{O}_F)$. If $v(\alpha) > -\frac{nq}{q-1}$ for all valuations v of F extending v_∞ , then $[\alpha]_n$ is a Taelman unit and $U(C^{\otimes n}/\mathcal{O}_F)$ is generated by such symbols over $\mathbb{F}[t]$.*

The strategy of the proof of Theorem 1.2 is to write down the exponential map of E in terms of its motive $\underline{M}(E)$. This idea is not new and goes back to Sinha's appendix [Sin97], or later by Taelman [Tae20] with an alternative approach. Our construction is yet different, and is rooted on the abstract theory of Anderson generating functions that we develop therein. As another distinction, we work with the general coefficient ring A , where the module of Kähler differentials plays a decisive role.

After a brief review of Anderson modules and homological preliminaries, we give the proof of the first Barsotti–Weil-type isomorphism in Section 2. In Section 3, we develop the theory of Anderson generating functions, extending previous work of the author and Maurishcat [GM21] for special functions. Given an Anderson module E over a Banach algebra B , a generating function for E is an element $\psi \in E(B\langle A \rangle)$ that satisfies

$$\text{for all } a \in A : \quad (1 \otimes \phi_a)\psi - (a \otimes 1)\psi \in E(B).$$

We show that, up to a differential form, each generating function can be described as the generating series of division tower, as considered by Anderson. The exponential map is recovered by comparing the residues of generating functions at the characteristic graph and at ∞ . Section 4 is devoted to the proof of Theorem 1.2 and Corollary 1.3.

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2 Generalities on Anderson modules

2.1 Recollection

Let R be an A -algebra via a structure morphism $\ell : A \rightarrow R$. For our purposes, an *Anderson A -module* E over R is a smooth and affine A -module scheme over R satisfying the following two properties:

- (And)₁ As an $\mathbb{F}$ -module scheme over R , E is Zariski-locally isomorphic to $\mathbb{G}_a^d$ for some $d > 0$;
- (And)₂ For all $a \in A$, $\partial\phi_a - \ell(a)$ acts nilpotently on $\text{Lie}_E(R)$, where ϕ_a denotes the action of a on E and $\partial\phi_a$ its induced action on $\text{Lie}_E(R)$.

To E , we associate its *motive* $M(E) := \text{Hom}_{\mathbb{F}}(E, \mathbb{G}_a)$; *i.e.* the group of $\mathbb{F}$ -module schemes homomorphisms over R from E to $\mathbb{G}_a$. It acquires an action of A by acting on E and an action of R by acting on $\text{End}(\mathbb{G}_a)$; both actions coincide on $\mathbb{F}$, making $M(E)$ an $A \otimes R$ -module¹.

Because $\text{End}(\mathbb{G}_a) = R[\tau]$, where τ corresponds to the Frobenius endomorphism of $\mathbb{G}_a$, the hypothesis (And)₁ makes $M(E)$ a finitely generated projective² left $R[\tau]$ -module of constant rank d . The action of τ can also be made linear, giving rise to an $A \otimes R$ -linear map

$$\tau_E : \tau^*M(E) \longrightarrow M(E)$$

whose cokernel, thanks to (And)₂, is killed by a power of the ideal $\mathfrak{j} := \ker(A \otimes R \rightarrow R)$.

¹Throughout, unlabelled tensor products are over $\mathbb{F}$.

²Note that this is false if one merely assumes instead of (And)₁ that E is a flat form of $\mathbb{G}_a^d$, as in [Har19].

Definition 2.1 (Abelian A -modules). We call E *abelian* if $M(E)$ is finite projective over $A \otimes R$ of constant rank; equivalently, if the pair $\underline{M}(E) := (M(E), \tau_E)$ defines an A -motive over R in the sense of [Gaz22, Definition 3.2].

The data of $\underline{M}(E)$ is sufficient to recover E . If S is an R -algebra, we have a canonical A -linear isomorphism [GM25, Proposition 3.2(1)]

$$E(S) \cong \text{Hom}_{R[\tau]}(M(E), S) \quad (2.1)$$

where S is treated as a left $R[\tau]$ -module via $\tau \cdot s := s^q$. The isomorphism (2.1) maps $e_S \in E(S)$ to $(m \in M(E) \mapsto m(e_S) \in S)$.

Notation 2.2 (Extended functor of points). Formula (2.1) allows to consider points of E not only over R -algebras, but also over left $R[\tau]$ -modules T by setting

$$E(T) := \text{Hom}_{R[\tau]}(M(E), T). \quad (2.2)$$

This perspective was first use in [CGL26] and we will use it constantly throughout. If T is a left $R[\tau]$ -module on which τ acts by zero, we shall write

$$\text{Lie}_E(T) := E(T).$$

This is consistent with the usual functor Lie_E . When we write $\text{Lie}_E(T)$ for an R -module T , it is tacitly assumed that we treat T as a left $R[\tau]$ -module where τ acts by zero.

We end this subsection by the following lemma, which is well-known in the abelian case, *e.g.* [GM25, Corollary 6.6].

Lemma 2.3. *Let M be an $A \otimes R[\tau]$ -module which is finitely generated over $R[\tau]$. Let $u \in A$ be nonconstant. Then $\text{Hom}_{R[u, \tau]}(M, A \otimes R) = (0)$. In particular, $\text{Hom}_{A \otimes R[\tau]}(M, A \otimes R) = (0)$.*

Proof. First, we equip A with the degree $\deg = -v_\infty$. Given a finite dimensional $\mathbb{F}$ -subvector space V of R , we extend $\deg$ to $A \otimes V$ in the classical way; *i.e.* by taking the coordinate-wise maximum of the degree with respect to a chosen basis of V over $\mathbb{F}$ (although it does not depend on such a choice). Then, we extend $\deg$ to $A \otimes R$ by writing R as $\text{colim}_{V \subset R} V$, indexed over finite dimensional $\mathbb{F}$ -vector spaces inside R . Clearly, we have

$$\begin{aligned} \text{for } a \in A, h \in A \otimes R, & \quad \deg(a \cdot h) = \deg(a) + \deg(h), \\ \text{for } h_1, h_2 \in A \otimes R, & \quad \deg(h_1 + h_2) \leq \max\{\deg(h_1), \deg(h_2)\}. \end{aligned}$$

Let $f \in \text{Hom}_{R[u, \tau]}(M, A \otimes R)$. We fix a finite family $(m_1, \dots, m_g)$ which generates M over $R[\tau]$ and set $h_i := f(m_i) \in A \otimes R$, for $i \in \{1, \dots, g\}$. If $f \neq 0$, let $j \in \{1, \dots, g\}$ be such that $\deg(h_j)$ is maximal. Because $um_j \in M$, there exists $p_i(\tau) \in R[\tau]$ such that

$$um_j = \sum_{i=1}^g p_i(\tau) m_i \quad \text{and thus} \quad uh_j = \sum_{i=1}^g p_i(\tau) h_i$$

But the left-hand side of the latter equality has degree $> \deg(h_j)$ while the right-hand side has degree $\leq \deg(h_j)$, a contradiction. Therefore $f = 0$. $\square$

2.2 Functor of A -invariants

We call $u \in A$ *separating* if $K/\mathbb{F}(u)$ is finite separable; equivalently, $\mathbb{F}[u] \hookrightarrow A$ is finite flat and generically étale. Given a separating element $u \in A$, we define the *dualizing module of $A/\mathbb{F}[u]$* :

$$\mathcal{D}_{A/\mathbb{F}[u]} := \{x \in A \otimes_{\mathbb{F}[u]} A \mid \text{for all } a \in A : (a \otimes_{\mathbb{F}[u]} 1 - 1 \otimes_{\mathbb{F}[u]} a)x = 0\}.$$

It is an A -module (with respect to the left or right multiplication by A , which coincide by construction). Under the multiplication map $A \otimes_{\mathbb{F}[u]} A \rightarrow A$ the module $\mathcal{D}_{A/\mathbb{F}[u]}$ corresponds to the different ideal $\mathfrak{d}_{A/\mathbb{F}[u]}$. From generalities on Kähler differentials (*e.g.* [GM21, §3.2]), we have a canonical isomorphism

$$\mathcal{D}_{A/\mathbb{F}[u]} \otimes_A \Omega_{A/\mathbb{F}}^1 \xrightarrow{\sim} Adu, \quad x \otimes_A \omega \mapsto x\omega. \quad (2.3)$$

For later use, we set

$$Du := \sum_i d_i \otimes_A \omega_i \in \mathcal{D}_{A/\mathbb{F}[u]} \otimes_A \Omega_{A/\mathbb{F}}^1, \quad d_i \in \mathcal{D}_{A/\mathbb{F}[u]}, \quad \omega_i \in \Omega_{A/\mathbb{F}}^1 \quad (2.4)$$

be the inverse image of du through (2.3) (*the dualizing element*).

More generally, we will require the *functor of A -invariants*.

Definition 2.4 (Functor of A -invariants). We call the functor

$$\begin{aligned} (-)^A : \{(A \otimes_{\mathbb{F}[u]} A) - \text{modules}\} &\longrightarrow \{A - \text{modules}\}, \\ V &\longmapsto V^A := \{v \in V \mid (a \otimes 1)v = (1 \otimes a)v\} \end{aligned}$$

which sends an $A \otimes_{\mathbb{F}[u]} A$ -module to its submodule on which both A -actions coincide, the *functor of A -invariants*.

This functor maps the unit module $A \otimes_{\mathbb{F}[u]} A$ to $\mathcal{D}_{A/\mathbb{F}[u]}$ which, as mentioned above, is isomorphic to the dual of $\Omega_{A/\mathbb{F}}^1$. More generally, we have:

Lemma 2.5. *Let T be an A -module. Then $(A \otimes_{\mathbb{F}[u]} T)^A = \mathcal{D}_{A/\mathbb{F}[u]} \otimes_A T$.*

Proof. Consider the A -module

$$\omega_{A/\mathbb{F}[u]} := \text{Hom}_{\mathbb{F}[u]}(A, \mathbb{F}[u]).$$

Since A is finite projective over B , evaluation induces, for every A -module T , a canonical isomorphism

$$A \otimes_{\mathbb{F}[u]} T \xrightarrow{\sim} \text{Hom}_{\mathbb{F}[u]}(\omega_{A/\mathbb{F}[u]}, T), \quad a \otimes t \mapsto (\lambda \mapsto \lambda(a)t).$$

Under this isomorphism, multiplication by $a \in A$ on the first factor of $A \otimes_{\mathbb{F}[u]} T$ corresponds to $f(\lambda) \mapsto f(a\lambda)$, whereas multiplication by a on T corresponds to $f(\lambda) \mapsto af(\lambda)$. Consequently, the relation

$$\text{for } v \in A \otimes_{\mathbb{F}[u]} T : \quad (a \otimes 1)v = (1 \otimes a)v \quad \text{for every } a \in A$$

is equivalent to A -linearity of the corresponding map $\omega_{A/\mathbb{F}[u]} \rightarrow T$. Therefore

$$(A \otimes_{\mathbb{F}[u]} T)^A \cong \text{Hom}_A(\omega_{A/\mathbb{F}[u]}, T).$$

Taking $T = A$ gives

$$\mathcal{D}_{A/\mathbb{F}[u]} = (A \otimes_{\mathbb{F}[u]} A)^A \cong \text{Hom}_A(\omega_{A/\mathbb{F}[u]}, A).$$

Moreover, $\omega_{A/\mathbb{F}[u]}$ is a finite torsion-free A -module of rank one. Since A is a Dedekind domain, $\omega_{A/\mathbb{F}[u]}$ is invertible. Hence

$$\text{Hom}_A(\omega_{A/\mathbb{F}[u]}, T) \cong \text{Hom}_A(\omega_{A/\mathbb{F}[u]}, A) \otimes_A T \cong \mathcal{D}_{A/\mathbb{F}[u]} \otimes_A T.$$

Combining the preceding isomorphisms yields

$$(A \otimes_{\mathbb{F}[u]} T)^A \cong \mathcal{D}_{A/\mathbb{F}[u]} \otimes_A T,$$

as claimed. □

Proposition 2.6. *Let M be an $A \otimes R[\tau]$ -module, finitely generated over $R[\tau]$. Then*

$$\mathrm{Ext}_{A \otimes R[\tau]}^1(M, A \otimes R) = \mathrm{Ext}_{R[u, \tau]}^1(M, A \otimes R)^A.$$

Proof. We begin with the formula $\mathrm{Hom}_{R[u, \tau]}(M, -)^A = \mathrm{Hom}_{A \otimes R[\tau]}(M, -)$, where both $(-)^A$ and $\mathrm{Hom}_{R[u, \tau]}(M, -)$ are additive, left-exact functors. We write $(-)^{R^i A}$ for the i th right derived functor of $(-)^A$. The standard hypothesis are satisfied to apply the Grothendieck spectral sequence theorem, and from the first-quadrant five terms exact sequence we get:

$$\mathrm{Hom}_{R[u, \tau]}(M, -)^{R^1 A} \rightarrow \mathrm{Ext}_{A \otimes R[\tau]}^1(M, -) \xrightarrow{d} \mathrm{Ext}_{R[u, \tau]}^1(M, -)^A \rightarrow \mathrm{Hom}_{R[u, \tau]}(M, -)^{R^2 A} \rightarrow \dots$$

where d is the canonical map. Applied to the left $A \otimes R[\tau]$ -module $A \otimes R$ and using Lemma 2.3, the first and fourth terms vanish and we obtain that d is an isomorphism. $\square$

2.3 Barsotti–Weil isomorphism

The Barsotti–Weil isomorphism relates the module of points of E to extension modules in categories of A -motives. To our knowledge, the strongest form of the statement is originally due to Taelman [Tae20, Theorem 8.1.1], generalizing prior work of Papanikolas and Ramachandran [PR03], and Woo [Woo95]. We also refer the reader to the recent work of Głoch–Kędzierski–Krasoń [GKK25, KK26, KP24] for further results in this direction. The formula was subsequently generalized for general coefficient rings A by Mornev [Mor18, Theorem 9.4]. For perfect A -algebras, it was obtained as a consequence of Hartl–Juschka’s pairing in [GM25]. We prove it in full generality here, without abelianness assumptions.

Theorem 2.7 (Barsotti–Weil isomorphism). *Let E be a (not necessarily abelian) Anderson A -module over an A -algebra R . There is an isomorphism of A -modules*

$$E(R) \cong \mathrm{Ext}_{A \otimes R[\tau]}^1(M(E), A \otimes R) \otimes_A \Omega_{A/\mathbb{F}}^1.$$

Proof. We start with the identification (2.1). Fix a separating element $u \in A$. Writing $M = M(E)$, we have

$$A \otimes E(R) = \mathrm{Hom}_{R[\tau]}(M, A \otimes R),$$

where $A \otimes R$ is treated as a left $R[\tau]$ -module with the A -linear action of τ . Hence

$$\begin{aligned} A \otimes_{\mathbb{F}[u]} E(R) &= \frac{A \otimes E(R)}{(u \otimes 1 - 1 \otimes \phi_u)(A \otimes E(R))} \\ &= \frac{\mathrm{Hom}_{R[\tau]}(M, A \otimes R)}{\{(u \otimes 1)g(*) - g((1 \otimes u)*) \mid g \in \mathrm{Hom}_{R[\tau]}(M, A \otimes R)\}} \end{aligned}$$

where the second isomorphism follows from the A -linearity of the identification (2.1). Using that M is projective over $R[\tau]$, we see that the right-hand side computes extensions of M by $A \otimes R$ in the category of $R[u, \tau]$ -modules (the underlying sequence of $R[\tau]$ -module splits, so the action of u is parametrized by an $R[\tau]$ -linear map from M to $A \otimes R$). We obtain an isomorphism

$$A \otimes_{\mathbb{F}[u]} E(R) \simeq \mathrm{Ext}_{R[u, \tau]}^1(M, A \otimes R). \quad (2.5)$$

Applying $(-)^A$ to (2.5) and appealing to Lemma 2.5 and Proposition 2.6, we get

$$\mathcal{D}_{A/\mathbb{F}[u]} \otimes_A E(R) \cong \mathrm{Ext}_{A \otimes R[\tau]}^1(M, A \otimes R),$$

which is the desired Barsotti–Weil formula after twisting by $\Omega_{A/\mathbb{F}}^1$. $\square$

Corollary 2.8. *Let E be an abelian Anderson A -module over R and let $\underline{M}(E)^\vee$ be the dual of its A -motive. Then, there is an isomorphism*

$$E(R) \cong \mathrm{Ext}_{\mathrm{AMot}_R}^{1,\mathrm{reg}}(\mathbb{1}, \underline{M}(E)^\vee) \otimes_A \Omega_{A/\mathbb{F}}^1.$$

Proof. The category of effective A -motives embeds fully and faithfully into the category of $A \otimes R[\tau]$ -modules and its image is closed under extensions. In particular,

$$\mathrm{Ext}_{A \otimes R[\tau]}^1(M(E), A \otimes R) \cong \mathrm{Ext}_{\mathrm{AMot}_R}^{1,\mathrm{eff}}(\underline{M}(E), \mathbb{1}).$$

where the superscript “eff” means extensions of effective A -motives. It is straightforward from [Gaz24, Proposition 5.3] that an extension of $\underline{M}(E)$ by $\mathbb{1}$ in AMot_R is effective if and only if it is regulated. Consequently, using [Gaz24, Corollary 5.4], we get

$$\mathrm{Ext}_{\mathrm{AMot}_R}^{1,\mathrm{eff}}(\underline{M}(E), \mathbb{1}) \cong \mathrm{Ext}_{\mathrm{AMot}_R}^{1,\mathrm{reg}}(\underline{M}(E), \mathbb{1}) \cong \mathrm{Ext}_{\mathrm{AMot}_R}^{1,\mathrm{reg}}(\mathbb{1}, \underline{M}(E)^\vee).$$

The desired statement therefore follows from Theorem 2.7. $\square$

Remark 2.9 (Computing the Barsotti–Weil isomorphism). In [Gaz24, Proposition 5.3], the module of effective/regulated extensions was rather presented as a quotient of $\mathrm{Hom}_{A \otimes R}(\tau^*M(E), A \otimes R)$ via a surjective map

$$\iota : \mathrm{Hom}_{A \otimes R}(\tau^*M(E), A \otimes R) \rightarrow \mathrm{Ext}_{A \otimes R[\tau]}^1(M(E), A \otimes R)$$

assigning to f the class of the extension $(A \otimes R) \oplus M(E)$ (which splits at the level of $A \otimes R$ -modules) where the action of τ is represented by 2-by-2 upper triangular matrix

$$\begin{pmatrix} \tau_{\mathbb{1}} & f \\ 0 & \tau_M \end{pmatrix}.$$

Here, we describe how to compute the Barsotti–Weil map under this presentation, in the case where $A = \mathbb{F}[t]$ (and u is chosen to be t).

Let e be an element in $E(R)$, seen as a morphism in $\mathrm{Hom}_{R[\tau]}(M(E), R)$. We deduce from the proof of Theorem 2.7 that the Barsotti–Weil isomorphism maps e to the extension $H = H_e$ of $M(E)$ by $R[t]$ given as follows: $H = R[t] \oplus M(E)$, where the action of τ , denoted by τ_H , is diagonal and where the action of t , denoted by t_H , is given by a 2-by-2 upper triangular matrix

$$t_H = \begin{pmatrix} t & e \\ 0 & t_{M(E)} \end{pmatrix}.$$

To describe H_e as arising from the image of ι , select an R -linear map $h : M(E) \rightarrow R[t]$ such that

$$e = t \circ h - h \circ t_M.$$

Consider the R -module $H'_e := R[t] \oplus M(E)$ equipped as an $R[t, \tau]$ -module, where $t_{H'}$ acts diagonally but where $\tau_{H'}$ is given by the 2-by-2 upper triangular matrix

$$\tau_{H'} = \begin{pmatrix} \tau_{\mathbb{1}} & h \circ \tau_{M(E)} - \tau_{\mathbb{1}} \circ \tau^*h \\ 0 & \tau_M \end{pmatrix}.$$

By design, H'_e is an extension of $M(E)$ by $R[t]$ as $R[t, \tau]$ modules, and $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} : H_e \rightarrow H'_e$ defines an $R[t, \tau]$ -linear equivalence of extensions. The equivalence class of H'_e is that of $\iota(h \circ \tau_{M(E)} - \tau_{\mathbb{1}} \circ \tau^*h)$. In conclusion, the Barsotti–Weil isomorphism maps $e \in \mathrm{Hom}_{R[\tau]}(M(E), R)$ to the image through ι of

$$h \circ \tau_{M(E)} - \tau_{\mathbb{1}} \circ \tau^*h \in M = \mathrm{Hom}_{R[t]}(\tau^*M(E), R[t])$$

where $h : M(E) \rightarrow R[t]$ is any R -linear map such that $e = t \circ h - h \circ t_{M(E)}$.

Remark 2.10 (The Barsotti-Weil isomorphism for Carlitz twists). We apply the recipe of Remark 2.9 to compute the (inverse of the) Barsotti-Weil isomorphism in the case of the Carlitz n th twist, where $n \geq 1$. Recall that the latter is the Anderson t -module $C^{\otimes n}$ whose S -points, for an R -algebra S , are given by the $\mathbb{F}$ -vector space S^n together with the action of t given by

$$\text{for } (x_1, \dots, x_n) \in S^n : \quad t \cdot (x_1, \dots, x_n) := (\theta x_1 + x_2, \dots, \theta x_{n-1} + x_n, \theta x_n + x_1^q),$$

where $\theta := \ell(t)$. Given $\alpha \in S$, we set $[\alpha]_n := (0, \dots, 0, -\alpha) \in C^{\otimes n}(S)$. We call $[\alpha]_n$ the *symbol* of α . It can be easily verified directly from the formula that $C^{\otimes n}(S)$ is generated by symbols as an $\mathbb{F}[t]$ -module, although the content of the present remark reproves this.

We denote by $\kappa_i : C^{\otimes n} \rightarrow \mathbb{G}_a$ the projection onto the i th coordinate. The module $M(C^{\otimes n})$ is free of rank one over $R[t]$, generated by the first projection κ_1 , and $\kappa_i = (t - \theta)^{i-1} \kappa_1$ for $i \in \{1, \dots, n\}$. In particular, an element $\alpha \in R$ also defines an element of $\text{Hom}_{R[t]} \tau^* M(C^{\otimes n}), R[t]$ by (semilinear) multiplication, and therefore an effective/regulated extension $\iota(\alpha)$ of $\underline{M}(C^{\otimes n})$ by $\mathbb{1}$. The extension $\iota(\alpha)$ was named *the polylog class* of α in [GM22]. We claim that the Barsotti-Weil isomorphism makes $[\alpha]_n$ correspond to $\iota(\alpha)$.

To prove this claim, consider the unique R -linear map $h : M(C^{\otimes n}) \rightarrow R[t]$ satisfying

1. for $j \in \{0, \dots, n-1\}$, $h((t - \theta)^j \kappa_1) = 0$,
2. for all $m \in M(C^{\otimes n})$, $h(\tau m) - \tau(h(m)) = \alpha \tau(m)$.

By Remark 2.9, the morphism $e := t \circ h - h \circ t_M$ defines an element in $\text{Hom}_{R[\tau]}(M(C^{\otimes n}), R[t])$, hence in $C^{\otimes n}(R)$, and we have to show that $e = [\alpha]_n$. We compute

$$\text{for } i \in \{1, \dots, n-1\} : \quad e(\kappa_i) = th((t - \theta)^{i-1} \kappa_1) - h(t(t - \theta)^{i-1} \kappa_1) = 0,$$

and for $i = n$,

$$\begin{aligned} e(\kappa_n) &= th((t - \theta)^{n-1} \kappa_1) - h(t(t - \theta)^{n-1} \kappa_1) \\ &= -h(t(t - \theta)^{n-1} \kappa_1) = -h((t - \theta)^n \kappa_1) + h(\theta(t - \theta)^{n-1} \kappa_1) \\ &= -h(\tau \kappa_1) \\ &= -\alpha. \end{aligned}$$

Hence $e(m) = m([\alpha]_n)$ for all $m \in M(C^{\otimes n})$, as desired.

Because $\text{Ext}_{\text{AMot}_R}^{1, \text{eff}}(\underline{M}(C^{\otimes n}), \mathbb{1}) = \text{Ext}_{\text{AMot}_R}^{1, \text{reg}}(\underline{M}(C^{\otimes n}), \mathbb{1})$ is generated as an $\mathbb{F}[t]$ -module by the extensions $\iota(\alpha)$ for $\alpha \in R$ (see [Gaz24, Proposition 5.3]), we deduce that $C^{\otimes n}(R)$ is generated by the symbols $[\alpha]_n$.

3 Anderson generating functions

Anderson generating functions were already used in [And86, §3] in order for Anderson to relate scattering matrices and uniformizability of t -modules. They seem to acquire their naming much later in the work of El-Guindy–Papanikolas [EP14]. The terminology “generating” evokes the construction of such series, which are built as generating series of t -division sequences of an Anderson t -module. In the setting of the Carlitz module, the archetype is the following series in t

$$f_C(z; t) := - \sum_{i=0}^{\infty} \exp_C \left(\frac{z}{\theta^{i+1}} \right) t^i,$$

where z is a parameter in $\text{Lie}_C(\mathbb{C}_{\infty}) = \mathbb{C}_{\infty}$. Using the t -linear exponential $\widehat{\exp}$, we can equivalently write

$$f_C(z; t) = \widehat{\exp}_C \left(\frac{z}{t - \theta} \right). \quad (3.1)$$

For $z = -\tilde{\pi}$, $f_C(z; t)$ recovers the Anderson–Thakur special function $\omega(t)$.

In [GM21, GM24], Maurischat and the author laid the foundations for a theory of special functions of an arbitrary Anderson module. In this section, we proceed likewise for generating functions, extending the results of *loc. cit.* We take the functional equation satisfied by generating functions as the defining property, resulting in an A -module of *generating functions*

$$\mathfrak{gf}_E(\mathbb{C}_\infty)$$

which contains special functions $\mathfrak{sf}_E(\mathbb{C}_\infty)$ as a submodule. Given a parameter $z \in \text{Lie}_E(\mathbb{C}_\infty)$, we rather associate an element $f_E(z)$ of $\mathfrak{gf}_E(\mathbb{C}_\infty) \otimes_A \Omega_{A/\mathbb{F}}^1$ which, in the case of Carlitz, recovers $f_C(z; t)dt$.

The general theory of generating functions will enable us to shed a new perspective on the following theorem, originally due to Sinha [Sin97, Theorem A.0.7] in the $\mathbb{F}[t]$ -setting:

Theorem 3.1. *Let B be a Banach F_∞ -algebra and let E be a (non-necessarily abelian) Anderson A -module over B . There is a commutative diagram*

$$\begin{array}{ccc} \text{Hom}_{A \otimes B[\tau]} \left(M(E), \frac{B\langle A \rangle}{A \otimes B} \right) \otimes_A \Omega_{A/\mathbb{F}}^1 & \xrightarrow{\delta} & \text{Ext}_{A \otimes B[\tau]}^1(M(E), A \otimes B) \otimes_A \Omega_{A/\mathbb{F}}^1 \\ \downarrow \wr & & \downarrow \wr \text{Thm 2.7} \\ \text{Lie}_E(B) & \xrightarrow{\exp_E} & E(B) \end{array} \quad (3.2)$$

where the top horizontal arrow is the connecting homomorphism obtained after applying the functor $\text{Hom}_{A \otimes B[\tau]}(M(E), -)$ to the short exact sequence $0 \rightarrow A \otimes B \rightarrow B\langle A \rangle \rightarrow \frac{B\langle A \rangle}{A \otimes B} \rightarrow 0$.

Theorem 3.1 is obtained by showing that the residue at $\mathfrak{j}$ of $f_E(z)$ is z while its “residue at ∞ ” is $-\exp_E(z)$ (we explain later what it means to take the residue at ∞ of such Tate series, hence the quotation marks). For the Carlitz module, this is recovered from the following formula borrowed from [EP14], which holds for small z :

$$f_C(z; t) = \sum_{i=0}^{\infty} \frac{\exp_C(z)^{q^i}}{(t - \theta)(t - \theta^q) \cdots (t - \theta^{q^i})}. \quad (3.3)$$

In our general setting, no comparably elegant formula seems to be available.

3.1 Preliminary notation

Let B be a F_∞ -Banach algebra and E be an Anderson module over B . Recall that the A -module $E(B)$ of B -points of E carries a metric by identifying it with the group $\text{Hom}_{B[\tau]}(M(E), B)$ and choosing a finite set of generators of $M(E)$ as a $B[\tau]$ -module. This metric depends on the choice of the generators, hence is not canonical. However, the set of null sequences in $E(B)$ does not depend on such choices. Consequently, we may define

$$E(B\langle A \rangle) := A \hat{\otimes} E(B)$$

as the completion of $A \otimes E(B)$ with respect to the zero sequences in $E(B)$ (see [GM21, §2]). This is consistent with the extended functor of points notation (2.2), where $B\langle A \rangle$ is treated as a left $B[\tau]$ -module where τ acts A -linearly:

$$E(B\langle A \rangle) \cong \text{Hom}_{B[\tau]}(M(E), B\langle A \rangle) \quad (3.4)$$

(see [GM21, Theorem 3] and its proof). In the sequel, we will also consider $E(\mathcal{A})$ where $\mathcal{A}$ is one of the following “functional” left $B[\tau]$ -module:

- *Classical Tate algebra.* Let $u \in A$ be non constant. We write $B\langle u \rangle$ for the Tate B -algebra in the variable u . It is a $B[\tau]$ -submodule of $B\langle A \rangle$.
- *Rings of entire functions.* We write $B\langle\langle u \rangle\rangle$ for the B -algebra of entire functions in u . More generally, the algebra of entire functions $B\langle\langle A \rangle\rangle$ is defined as the limit $\lim_{\rho>1} B\langle A \rangle_\rho$ (see [GM24, §3] for the notations); note that $\tau(B\langle A \rangle_\rho) \subset B\langle A \rangle_{\rho^q}$ so $B\langle A \rangle_\rho$ itself is not $B[\tau]$ -submodule of $B\langle A \rangle$ unless $\rho = 1$.
- *Module of meromorphic functions with poles of bounded order along j and iterates.* Consider the expression

$$\Pi_u := \prod_{i \geq 0} \left(1 - \frac{u \otimes 1}{1 \otimes u^{q^i}} \right)$$

which converges in $B\langle\langle u \rangle\rangle$, hence in $B\langle A \rangle_\rho$ for all $\rho > 0$, and thus in $B\langle\langle A \rangle\rangle$. Then, given a nonnegative integer n , the group $\Pi_u^{-n} B\langle\langle A \rangle\rangle$ is a $B[\tau]$ -submodule of $B\langle A \rangle$. Taking the intersection over all non-constant u , we define

$$B\langle\langle A \rangle\rangle_{(j)} := \bigcup_{n \geq 0} \bigcap_{u \in A \setminus \mathbb{F}} \Pi_u^{-n} B\langle\langle A \rangle\rangle$$

which is also a $B[\tau]$ -submodule of $B\langle A \rangle$; it consists of meromorphic functions with poles of bounded order at j and its iterates under the A -linear Frobenius.

If $\mathcal{A}$ is a $A \otimes B[\tau]$ -module, we will often use that $E(\mathcal{A} \otimes_A \Omega_{A/\mathbb{F}}^1)$ is canonically isomorphic to $E(\mathcal{A}) \otimes_A \Omega_{A/\mathbb{F}}^1$. For the latter, we point out that the tensor product is taken with respect to the A -module structure on $\mathcal{A}$ (on not the one on E).

Construction 3.2 (Residue at j of E -valued functions). Consider the B -linear map

$$\text{res}_j : B\langle\langle A \rangle\rangle_{(j)} \otimes_A \Omega_{A/\mathbb{F}}^1 \longrightarrow B \quad (3.5)$$

which extracts the residue at j . Note that if a meromorphic differential in the source can be written as $\tau(\eta)$, then $\tau(\eta)$ is regular at j as otherwise η would have a pole at $j^{(-1)}$ which is prescribed being an element of $B\langle\langle A \rangle\rangle_{(j)} \otimes_A \Omega_{A/\mathbb{F}}^1$. In particular, $\text{res}_j(\tau(\eta)) = 0$. We deduce that (3.5) promotes to a map of $B[\tau]$ -modules, where τ acts by zero on its target B . Applying the extended functor of points E of notations (2.2), we obtain a map

$$\text{res}_j : E(B\langle\langle A \rangle\rangle_{(j)}) \otimes_A \Omega_{A/\mathbb{F}}^1 \longrightarrow \text{Lie}_E(B). \quad (3.6)$$

The map (3.6) is the “residue at j of E -valued functions” considered in [GM24, §5.1.2].

3.2 Analytic generating functions

By analogy with the module of special functions defined in [GM21], we introduce the following definition.

Definition 3.3 (Analytic generating functions). The module of *Anderson generating functions* of E over B is the A -module

$$\mathfrak{gf}_E(B) := \{ \Psi \in E(B\langle A \rangle) \mid \text{for all } a \in A : (a \otimes 1 - 1 \otimes \phi_a) \Psi \in A \otimes E(B) \}.$$

Remark 3.4. We distinguish two canonical submodules:

- The module of *special functions* $\mathfrak{sf}_E(B)$ of [GM21, GM24] is the submodule of elements $\omega \in E(B\langle A \rangle)$ such that $(a \otimes 1 - 1 \otimes \phi_a) \omega = 0$ for all $a \in A$.

- $A \otimes E(B)$ is a submodule of $\mathfrak{gf}_E(B)$.

Let $u \in A$ be a separating element and let $z \in \text{Lie}_E(B)$. We consider the following expression

$$f_E(z; u) := \widehat{\exp}_E((u \otimes 1 - 1 \otimes \partial\phi_u)^{-1}z) \in E(B\langle u \rangle)$$

which are similar to those originally considered by Anderson, and generalizes the ones in the Carlitz module case (3.1). The element $f_E(z; u)$ is well-defined because $u \otimes 1 - 1 \otimes \ell(u)$ as an element in $B\langle A \rangle$, e.g. [Gaz24, Lemma 3.15], and hence $u \otimes 1 - 1 \otimes \partial\phi_u$ as an endomorphism of $\text{Lie}_E(B\langle A \rangle)$, is invertible. In addition, it satisfies the relation

$$(u \otimes 1 - 1 \otimes \phi_u)f_E(z; u) = \exp_E(z),$$

from which we deduce that $(a \otimes 1 - 1 \otimes \phi_a)f_E(z; u) \in A \otimes E(B)$ for all $a \in \mathbb{F}[u]$. The following series expansion is derived as in [GM24, Lemma 5.7].

Lemma 3.5. *Fix a choice of coordinates $\kappa : E \xrightarrow{\sim} \mathbb{G}_{a,B}^d$ and denote by $(e_i^\kappa)_{i \geq 0} \in \text{Mat}_d(B)$ the coefficients of $\exp_E$ under κ . Write $\text{Lie}_\kappa : \text{Lie}_E(B) \rightarrow B^d$ for the induced map of Lie spaces. We have*

$$f_E(z; u) = \sum_{j=0}^{d-1} \sum_{i \geq 0} \frac{(1 \otimes e_i^\kappa)[(1 \otimes \text{Lie}_\kappa)(1 \otimes n_u^j)(z)]^{(i)}}{(u \otimes 1 - 1 \otimes \ell(u)q^i)^{j+1}} \quad (3.7)$$

where n_u is the nilpotent endomorphism $\partial\phi_u - \ell(u)$ of $\text{Lie}_E(B)$. In particular, $f_E(z; u)$ belongs to $E(\Pi_u^{-d}B\langle u \rangle)$.

In general, the inclusion $\mathbb{F}[u] \subset A$ is strict and $f_E(z; u)$ is not yet an element in $\mathfrak{gf}_E(B)$. However, appealing to the dualizing element of (2.4), we can apply an “average transformation” to derive one up to a twist by the module of Kähler differentials. This is the content of the next proposition. To state it, recall that we denoted by

$$\sum_i d_i \otimes_A \omega_i \in \mathcal{D}_{A/\mathbb{F}[u]} \otimes_A \Omega_{A/\mathbb{F}}^1$$

the dualizing element of (2.4). Further choose representation of d_i as $\sum_j a_{ij} \otimes b_{ij} \in A \otimes A$.

Proposition 3.6 (Constructing all generating functions). *Consider the expression*

$$f_E(z) := \sum_{i,j} a_{i,j} \omega_i \otimes \phi_{b_{i,j}} f_E(z; u) \in E(\Omega_{A/\mathbb{F}}^1 \otimes_A B\langle A \rangle).$$

Then,

(i) $f_E(z)$ belongs to $\mathfrak{gf}_E(B) \otimes_A \Omega_{A/\mathbb{F}}^1$,

(ii) The map

$$\text{Lie}_E(B) \longrightarrow \frac{\mathfrak{gf}_E(B)}{A \otimes E(B)} \otimes_A \Omega_{A/\mathbb{F}}^1, \quad z \longmapsto f_E(z) + \Omega_{A/\mathbb{F}}^1 \otimes E(B) \quad (3.8)$$

is an A -linear isomorphism.

(iii) The inverse of (3.8) is obtained by taking the residue at $\mathfrak{j}$. In particular, (3.8) does not depend on the choice of u .

Proof of Proposition 3.6. Consider the $\mathbb{F}[u]$ -linear map

$$f_E(-; u) : \text{Lie}_E(B) \longrightarrow \frac{\{\psi \in E(B\langle u \rangle) \mid (u \otimes 1 - 1 \otimes \phi_u)\psi \in \mathbb{F}[u] \otimes_{\mathbb{F}} E(B)\}}{\mathbb{F}[u] \otimes E(B)}$$

which assigns to $z \in \text{Lie}_E(B)$ the expression $f_E(z; u)$ modulo $\mathbb{F}[u] \otimes E(B)$. To show that this map is surjective, let y be an element in the target. Because $\mathbb{F}[u] \otimes E(B)$ is dense in $E(B\langle u \rangle)$, up to subtracting an element of $\mathbb{F}[u] \otimes E(B)$ to y we may assume that y belongs to a ball of small enough radius in $E(B\langle u \rangle)$ on which the $\mathbb{F}[u]$ -linear exponential $\widehat{\exp}_E$ is an isometry. Assuming so, we can write $y = \widehat{\exp}_E(g)$ for some $g \in \text{Lie}_E(B\langle A \rangle)$. From the relation $(u \otimes 1 - 1 \otimes \phi_u)y \in \mathbb{F}[u] \otimes E(B)$, we deduce that $(u \otimes 1 - 1 \otimes \partial\phi_u)g \in \mathbb{F}[u] \otimes \Lambda_E + \mathbb{F}[u] \otimes \text{Lie}_E(B)$. But as $\mathbb{F}[u] \otimes \Lambda_E$ is discrete in $\mathbb{F}[u] \otimes \text{Lie}_E(B)$, up to shrinking the ball if necessary, we may assume that $(u \otimes 1 - 1 \otimes \partial\phi_u)g \in \mathbb{F}[u] \otimes \text{Lie}_E(B)$. We deduce that g satisfies

$$(u \otimes 1 - 1 \otimes \partial\phi_u)g = \sum_i p_i(u) \otimes z_i, \quad \text{and} \quad y = \widehat{\exp}_E \left((u \otimes 1 - 1 \otimes \partial\phi_u)^{-1} \sum_i p_i(u) \otimes z_i \right).$$

Consequently,

$$\begin{aligned} y &= \widehat{\exp}_E \left((u \otimes 1 - 1 \otimes \partial\phi_u)^{-1} \sum_i p_i(u) \otimes z_i \right) \\ &\equiv \widehat{\exp}_E \left((u \otimes 1 - 1 \otimes \partial\phi_u)^{-1} \sum_i 1 \otimes \partial\phi_{p_i(u)}(z_i) \right) \pmod{\mathbb{F}[u] \otimes E(B)} \\ &= f_E \left(\sum_i \partial\phi_{p_i(u)}(z_i); u \right) \pmod{\mathbb{F}[u] \otimes E(B)}. \end{aligned}$$

This shows that $f_E(-; u)$ is surjective. It is also injective because one deduces from (3.7) that the residue at the point $u \otimes 1 = 1 \otimes u$ of the differential $f_E(z; u)du$ is z .

Extending scalars in $f_E(-; u)$ along $\mathbb{F}[u] \rightarrow A$, we obtain an isomorphism

$$A \otimes_{\mathbb{F}[u]} \text{Lie}_E(B) \xrightarrow{\sim} \frac{\{\psi \in E(B\langle A \rangle) \mid (u \otimes 1 - 1 \otimes \phi_u)\psi \in A \otimes_{\mathbb{F}} E(B)\}}{A \otimes E(B)} \quad (3.9)$$

Applying the functor of A -invariants of Definition 2.4, we get

$$\mathcal{D}_{A/\mathbb{F}[u]} \otimes_A \text{Lie}_E(B) \xrightarrow{\sim} \frac{\mathfrak{gf}_E(B)}{A \otimes E(B)}$$

Tensoring over A with $\Omega_{A/\mathbb{F}}^1$ and using the isomorphism (2.3), we get

$$\text{Lie}_E(B) \xrightarrow{\sim} \frac{\mathfrak{gf}_E(B)}{A \otimes E(B)} \otimes_A \Omega_{A/\mathbb{F}}^1.$$

One checks by hand that the above maps z to $f_E(z) + \Omega_{A/\mathbb{F}}^1 \otimes E(B)$, proving at once (i) and (ii). To prove (iii), we write

$$\begin{aligned} f_E(z) &= \sum_{i,j} (a_{ij}\omega_i \otimes \phi_{b_{ij}}) \widehat{\exp}_E((u \otimes 1 - 1 \otimes \partial\phi_u)^{-1}z) \\ &= \sum_{i,j} (a_{ij}b_{ij}\omega_i \otimes 1) \widehat{\exp}_E((u \otimes 1 - 1 \otimes \partial\phi_u)^{-1}z) \\ &\quad - \sum_{i,j} (a_{ij}\omega_i \otimes 1) \widehat{\exp}_E((b_{ij} \otimes 1 - 1 \otimes \partial\phi_{b_{ij}})(u \otimes 1 - 1 \otimes \partial\phi_u)^{-1}z) \\ &= \widehat{\exp}_E((u \otimes 1 - 1 \otimes \partial\phi_u)^{-1}z) d(u \otimes 1) + (\text{some expression regular at } j). \end{aligned}$$

In particular, only the first term computes the residue at j . Because $u \in A$ is separating, $u \otimes 1 - 1 \otimes u$ is a uniformizing parameter at j (cf. [GM24, Lemma 2.3]). In the formula (3.7) of Lemma 3.5, the only pair of indices that contributes to the residue is $(i, j) = (0, 0)$ and we compute

$$\text{res}_j(f_E(z)) = \text{res}_j \widehat{\exp_E}((u \otimes 1 - 1 \otimes \partial \phi_u)^{-1} z) d(u \otimes 1) = \text{res}_j z \frac{d(u \otimes 1 - 1 \otimes u)}{u \otimes 1 - 1 \otimes u} = z.$$

□

From Proposition 3.6, one may compute the residue at j of generating functions. We now turn to the residue at ∞ .

Construction 3.7 (residue at ∞). We construct an A -linear map

$$-\text{res}_\infty : \frac{\mathfrak{g}f_E(B)}{A \otimes E(B)} \otimes_A \Omega_{A/\mathbb{F}}^1 \longrightarrow E(B) \quad (3.10)$$

that we will interpret later in Subsection 3.3 as taking (minus) “the residue at ∞ ” (hence the notation). We begin with the assignment $\psi \mapsto (u \otimes 1 - 1 \otimes \phi_u)\psi$ which gives an $A \otimes A$ -linear map

$$\{\psi \in E(B\langle A \rangle) \mid (u \otimes 1 - 1 \otimes \phi_u)\psi \in A \otimes E(B)\} \longrightarrow A \otimes E(B)$$

mapping $f_E(z; u)$ to $\exp_E(z)$. In addition, it induces an $A \otimes A$ -linear map:

$$\frac{\{\psi \in E(B\langle A \rangle) \mid (u \otimes 1 - 1 \otimes \phi_u)\psi \in A \otimes E(B)\}}{A \otimes E(B)} \rightarrow \frac{A \otimes E(B)}{(u \otimes 1 - 1 \otimes \phi_u)(A \otimes E(B))} \cong A \otimes_{\mathbb{F}[u]} E(B).$$

Applying the functor of A -invariants and twisting by $\Omega_{A/\mathbb{F}}^1$, we obtain a morphism as in (3.10).

The following proposition, which is the key to Sinha’s Theorem 3.1, is immediate from the construction.

Proposition 3.8. *The morphism of Construction 3.7 is the unique A -linear map*

$$\frac{\mathfrak{g}f_E(B)}{A \otimes E(B)} \otimes_A \Omega_{A/\mathbb{F}}^1 \longrightarrow E(B) \quad (3.11)$$

mapping $f_E(z)$ to $\exp_E(z)$. In particular, we have a commutative diagram:

$$\begin{array}{ccc} & \frac{\mathfrak{g}f_E(B)}{A \otimes E(B)} \otimes_A \Omega_{A/\mathbb{F}}^1 & \\ \text{res}_j \swarrow \sim & & \searrow -\text{res}_\infty \\ \text{Lie}_E(B) & \xrightarrow{\exp_E} & E(B) \end{array}$$

3.3 Formal generating functions

In this section, we justify why Construction 3.7 can be interpreted as taking the residue at ∞ . Let K_∞ denote the complete local field of $\mathbb{C}$ at ∞ ; we denote by $\mathcal{O}_\infty$ its valuation ring and by $\mathfrak{m}_\infty$ its maximal ideal. As in the theory of isocrystals, we consider the rings:

$$\mathcal{A}_\infty(B) := (\mathcal{O}_\infty \otimes B)_{\mathfrak{m}_\infty}^\wedge, \quad \mathcal{B}_\infty(B) := K_\infty \otimes_{\mathcal{O}_\infty} \mathcal{A}_\infty(B).$$

The ring $\mathcal{B}_\infty(B)$ allows to define the module of *formal generating functions*.

Definition 3.9 (Formal generating functions). The module of formal generating functions of E over B is the A -module

$$\mathfrak{GF}_E(B) := \{\psi \in E(\mathcal{B}_\infty(B)) \mid \text{for all } a \in A : (a \otimes 1 - 1 \otimes \phi_a)\psi \in A \otimes E(B)\}.$$

For elements in the module of formal generating functions it makes sense to consider the residue at ∞ . We consider the map $\text{res}_\infty : E(\mathcal{B}_\infty(B)) \otimes_A \Omega_{A/\mathbb{F}}^1 \rightarrow E(B)$ induced by taking the residue at ∞ $\mathcal{B}_\infty(B) \otimes_A \Omega_{A/\mathbb{F}}^1 \rightarrow B$ on $\text{Hom}_{B[\tau]}(M, -)$ (observe indeed that taking the residue at ∞ is a $B[\tau]$ -linear operation).

Proposition 3.10. *The residue at ∞ induces an A -linear isomorphism*

$$\frac{\mathfrak{GF}_E(B)}{A \otimes E(B)} \otimes_A \Omega_{A/\mathbb{F}}^1 \xrightarrow{\sim} E(B) \quad (3.12)$$

Proof. We first build a map ε as in (3.10) using the same method as in Construction 3.7, and then show that it coincides with taking (minus) the residue at ∞ . The former step is similar; given a separating element $u \in A$, we consider the canonical map

$$u \otimes 1 - 1 \otimes \phi_u : \{\psi \in E(\mathcal{B}_\infty(B)) \mid (u \otimes 1 - 1 \otimes \phi_u)\psi \in A \otimes E(B)\} \longrightarrow A \otimes E(B). \quad (3.13)$$

It is an isomorphism as, given $e \in A \otimes E(B)$, the following series converges in $E(\mathcal{B}_\infty(B))$

$$\xi_u(e) := \sum_{n \geq 0} (u^{-(n+1)} \otimes \phi_u^n)(e)$$

to an element that satisfies $(u \otimes 1 - 1 \otimes \phi_u)\xi_u(e) = e$. It is easily verified that $e \mapsto \xi_u(e)$ is an inverse for (3.13).

The map (3.13) induces a morphism between quotient modules where on the source one mods out by $A \otimes E(B)$ and, on the target, by $(u \otimes 1 - 1 \otimes \phi_u)(A \otimes E(B))$. We then twist by $\Omega_{A/\mathbb{F}}^1$ and apply the functor of A -invariants to get the map ε .

Explicitly, the inverse of ε maps e to $\Xi(e) := \sum_{i,j} (a_{ij}\omega_i \otimes \phi_{b_{ij}})\xi_u(e)$. We compute its residue at ∞ :

$$\text{res}_\infty(\Xi(e)) = \sum_{\substack{n \geq 0 \\ i,j}} \text{res}_\infty(a_{ij}\omega_i u^{-(n+1)} \otimes \phi_{b_{ij}} u^n(e)).$$

Note that $\text{res}_\infty(a_{ij}\omega_i u^{-(n+1)}) \in \mathbb{F}$ and vanishes for large $n \geq 0$. In particular, the above sum is finite and the following expression

$$S := \sum_{\substack{n \geq 0 \\ i,j}} \text{res}_\infty(a_{ij}\omega_i u^{-(n+1)}) u^n b_{ij}$$

belongs to A . On the other hand, S computes the residue at ∞ of the differential η of Lemma 3.11, and hence $S = -1$ by (iv). Consequently,

$$\text{res}_\infty(\Xi(e)) = \phi_S(e) = -e.$$

□

The following lemma was invoked in the proof.

Lemma 3.11. *For all i , fix a lift $\sum_j a_{ij} \otimes b_{ij} \in A \otimes A$ of $d_i \in \mathcal{D}_{A/\mathbb{F}[u]} \subset A \otimes_{\mathbb{F}[u]} A$ and consider the meromorphic differential*

$$\eta := \sum_{i,j} \frac{a_{ij}\omega_i \otimes b_{ij}}{u \otimes 1 - 1 \otimes u}$$

on the K^s -curve $C_{K^s} := C \times_{\mathbb{F}} K^s$ where K^s denotes a separable extension of K . Then

- (i) The class of η modulo $\Omega_{A/\mathbb{F}}^1 \otimes K^s$ does not depend on the lifts of d_i ;
- (ii) The residue of η at $\mathfrak{j}$ is 1;
- (iii) η is regular at $\mathfrak{j}^\sigma$ where σ is any $\mathbb{F}(u)$ -linear embedding $\sigma : K \rightarrow K^s$ different from the inclusion;
- (iv) The residue of η at ∞ is -1 .

Proof. Two lifts of d_i differ by a sum of elements of the form $ap(u) \otimes b - a \otimes p(u)b = (a \otimes b)(p(u) \otimes 1 - 1 \otimes p(u))$ for $a, b \in A$, $p(u) \in \mathbb{F}[u]$. The difference is then a multiple of $u \otimes 1 - 1 \otimes u$ which proves (i). Note that, because u is separating, the ideal $u \otimes 1 - 1 \otimes u$ decomposes as $\prod_\sigma \mathfrak{j}^\sigma$ in the Dedekind domain $A \otimes K^s$, where the product is indexed over the $\mathbb{F}(u)$ -linear embeddings $\sigma : K \rightarrow K^s$ (see Lemma 2.4 and its proof in [GM24]). In particular, $u \otimes 1 - 1 \otimes u$ is a uniformizing parameter at $\mathfrak{j}$. By writing

$$\eta = \sum_{i,j} \frac{a_{ij}\omega_i b_{ij} \otimes 1}{u \otimes 1 - 1 \otimes u} - \sum_{i,j} (a_{ij}\omega_i \otimes 1) \frac{b_{ij} \otimes 1 - 1 \otimes b_{ij}}{u \otimes 1 - 1 \otimes u}$$

we see that the former summand is $\frac{d(u \otimes 1)}{u \otimes 1 - 1 \otimes u}$ and the latter is regular at $\mathfrak{j}$; because $d(u \otimes 1) = d(u \otimes 1 - 1 \otimes u)$ in $\Omega_{A/\mathbb{F}}^1 \otimes K^s = \Omega_{A \otimes K^s / K^s}^1$, we get (ii). To prove (iii), we recall first that $\mathcal{D}_{A/\mathbb{F}[u]}$ is generated as an A -module by the elements $g_s \in A \otimes_{\mathbb{F}[u]} A$ constructed as follows:

1. $s \in A$ is an element such that $K = \mathbb{F}(u, s)$,
2. $f(X) \in \mathbb{F}[u][X]$ is the minimal polynomial of s over $\mathbb{F}(u)$,
3. $g(X, Y) := \frac{f(X) - f(Y)}{X - Y} \in \mathbb{F}[u][X, Y]$,
4. $g_s := g(s \otimes 1, 1 \otimes s)$.

Fix an $\mathbb{F}(u)$ -linear embedding $\sigma : K \rightarrow K^s$ which differs from the inclusion. As $f(s) = f(\sigma(s)) = 0$ but $s \neq \sigma(s)$, we see that g_s —and hence any element of $\mathcal{D}_{A/\mathbb{F}[u]}$ —belongs to the kernel of the map $A \otimes_{\mathbb{F}[u]} A \rightarrow K^s$, $a \otimes_{\mathbb{F}[u]} b \mapsto a\sigma(b)$. By writing

$$(u \otimes 1 - 1 \otimes u)\eta = \sum_{i,j} a_{ij}\omega_i \otimes b_{ij}$$

we deduce that the right-hand side vanishes modulo $\mathfrak{j}^\sigma$ whenever σ differs from the inclusion. In particular η is regular at $\mathfrak{j}^\sigma$. It remains to prove (iv): it follows from (iii) that $\mathfrak{j}$ and ∞ are the only possible poles of η . Using that the sum of the residues of a meromorphic differential on C_{K^s} vanishes, we deduce it from (ii). $\square$

Remark 3.12. We finally explain why Construction 3.7 can be thought of as taking the residue at ∞ . There is an abstract map

$$\frac{\mathfrak{gf}_E(B)}{A \otimes E(B)} \otimes_A \Omega_{A/\mathbb{F}}^1 \longrightarrow \frac{\mathfrak{G}\mathfrak{f}_E(B)}{A \otimes E(B)} \otimes_A \Omega_{A/\mathbb{F}}^1 \quad (3.14)$$

obtained by taking the minus the composition of (3.10) with the inverse of (3.12). Interpreting (3.14) as a “formalization map” implies that (3.10) should be thought as “minus (3.12)”, hence minus taking the residue at ∞ .

3.4 Exponential map as a connecting homomorphism

We are now in a position to prove Sinha's theorem. We begin with a proposition.

Proposition 3.13. *There is a commutative diagram*

$$\begin{array}{ccc} \frac{\mathfrak{g}f_E(B)}{A \otimes E(B)} \otimes_A \Omega_{A/\mathbb{F}}^1 & \xrightarrow{\sim} & \text{Hom}_{A \otimes B[\tau]} \left(M(E), \frac{B\langle A \rangle}{A \otimes B} \right) \otimes_A \Omega_{A/\mathbb{F}}^1 \\ \downarrow -\text{res}_\infty & & \downarrow \delta \\ E(B) & \xrightarrow[\text{Thm 2.7}]{\sim} & \text{Ext}_{A \otimes B[\tau]}^1(M(E), A \otimes B) \otimes_A \Omega_{A/\mathbb{F}}^1 \end{array}$$

where δ is the connecting homomorphism obtained after applying the functor $\text{Hom}_{A \otimes B[\tau]}(M(E), -)$ to the short exact sequence

$$0 \longrightarrow A \otimes B \longrightarrow B\langle A \rangle \longrightarrow \frac{B\langle A \rangle}{A \otimes B} \longrightarrow 0. \quad (3.15)$$

Proof. We already mentioned that the identification $E(B) = \text{Hom}_{B[\tau]}(M(E), B)$ promotes to $A \otimes E(B) = \text{Hom}_{B[\tau]}(M(E), A \otimes B)$ and $E(B\langle A \rangle) = \text{Hom}_{B[\tau]}(M(E), B\langle A \rangle)$ as $A \otimes A$ -modules. Because $M(E)$ is projective over $B[\tau]$, we also get

$$\frac{E(B\langle A \rangle)}{A \otimes E(B)} \xrightarrow{\sim} \text{Hom}_{B[\tau]} \left(M(E), \frac{B\langle A \rangle}{A \otimes B} \right). \quad (3.16)$$

Let $u \in A$ be a separating element. We identify on both sides of (3.16) the submodule of elements that are killed by $u \otimes 1 - 1 \otimes u$. This results in

$$\frac{\{\psi \in E(B\langle A \rangle) \mid (u \otimes 1 - 1 \otimes \phi_u)\psi \in A \otimes E(B)\}}{A \otimes E(B)} \xrightarrow{\sim} \text{Hom}_{B[u, \tau]} \left(M(E), \frac{B\langle A \rangle}{A \otimes B} \right). \quad (3.17)$$

Multiplication by $u \otimes 1 - 1 \otimes u$ maps the source to $A \otimes_{\mathbb{F}[u]} E(B)$ (identified with the cokernel of $u \otimes 1 - 1 \otimes \phi_u$ on $A \otimes E(B)$) and the target to

$$\frac{\text{Hom}_{B[\tau]}(M(E), A \otimes B)}{\{(u \otimes 1)g - g((1 \otimes u) \cdot) \mid g \in \text{Hom}_{B[\tau]}(M(E), A \otimes B)\}}. \quad (3.18)$$

The module (3.18) computes extensions between $M(E)$ and $A \otimes B$ in the category of $B[u, \tau]$ -modules. We obtain a commutative square

$$\begin{array}{ccc} \frac{\{\psi \in E(B\langle A \rangle) \mid (u \otimes 1 - 1 \otimes \phi_u)\psi \in A \otimes E(B)\}}{A \otimes E(B)} & \xrightarrow{\sim} & \text{Hom}_{B[u, \tau]} \left(M(E), \frac{B\langle A \rangle}{A \otimes B} \right) \\ \downarrow & & \downarrow \\ A \otimes_{\mathbb{F}[u]} E(B) & \longrightarrow & \text{Ext}_{B[u, \tau]}^1(M(E), A \otimes B) \end{array} \quad (3.19)$$

where the lower map is induced from $A \otimes_{\mathbb{F}[u]} E(B) = \text{Hom}_{B[\tau]}(M(E), A \otimes B) \rightarrow (3.18)$. To make $\mathfrak{g}f_E(B)$ appear in the upper left corner and rather get extensions of $A \otimes B[\tau]$ -modules in the lower right corner, we apply the functor of A -invariants. We obtain the square of A -modules

$$\begin{array}{ccc} \frac{\mathfrak{g}f_E(B)}{A \otimes E(B)} & \xrightarrow{\sim} & \text{Hom}_{A \otimes B[\tau]} \left(M(E), \frac{B\langle A \rangle}{A \otimes B} \right) \\ \downarrow & & \downarrow \\ \mathcal{D}_{A/\mathbb{F}[u]} \otimes_A E(B) & \longrightarrow & \text{Ext}_{A \otimes B[\tau]}^1(M(E), A \otimes B) \end{array} \quad (3.20)$$

which is the claimed diagram after twisting (3.20) by $\Omega_{A/\mathbb{F}}^1$. It is routine from the proof of Theorem 2.7 that the lower horizontal map is the Barsotti-Weil isomorphism and that the right-hand vertical map is δ . $\square$

Proof of Theorem 3.1. This follows immediately from Propositions 3.8 and 3.13. $\square$

4 Proof of the main theorem

Let E be an abelian and uniformizable Anderson module over $\mathcal{O}_F$ and $\underline{M}(E)$ be its motive. We write $\underline{M} = \underline{M}(E)^\vee$ for the dual of $\underline{M}(E)$ taken in the category of A -motives over $\mathcal{O}_F$. This final section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. As before, we write $\underline{M} = \underline{M}(E)^\vee$ where E is an abelian Anderson A -module over $\mathcal{O}_F$ which is uniformizable (over $F \otimes_K \mathbb{C}_\infty$). Given a flat $\mathcal{O}_F$ -algebra R and a K_∞ -Banach algebra B which contains R , we consider the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M \otimes_{\mathcal{O}_F} R & \longrightarrow & M \otimes_{A \otimes \mathcal{O}_F} B\langle A \rangle & \longrightarrow & M \otimes_{A \otimes \mathcal{O}_F} \frac{B\langle A \rangle}{A \otimes R} \longrightarrow 0 \\ & & \downarrow \text{id} - \tau_M & & \downarrow \text{id} - \tau_M & & \downarrow \text{id} - \tau_M \\ 0 & \longrightarrow & N \otimes_{\mathcal{O}_F} R & \longrightarrow & N \otimes_{A \otimes \mathcal{O}_F} B\langle A \rangle & \longrightarrow & N \otimes_{A \otimes \mathcal{O}_F} \frac{B\langle A \rangle}{A \otimes R} \longrightarrow 0 \end{array} \quad (4.1)$$

where $N = M + \tau_M(\tau^*M)$. Note that both rows are exact; for the top one, this follows by tensoring the exact sequence of $A \otimes \mathcal{O}_F$ -modules

$$0 \longrightarrow A \otimes R \longrightarrow B\langle A \rangle \longrightarrow \frac{B\langle A \rangle}{A \otimes R} \longrightarrow 0 \quad (4.2)$$

with the flat module M . The bottom line requires an additional argument to show that $N \otimes_{\mathcal{O}_F} R \rightarrow N \otimes_{A \otimes \mathcal{O}_F} B\langle A \rangle$ is injective as N may not be flat. Note that in $M[j^{-1}]$ we have an inclusion $N \subset j^{-n}M$ for n large enough. Since $\mathcal{O}_F \rightarrow R$ is flat, this further implies that $N \otimes_{\mathcal{O}_F} R \rightarrow j^{-n}(M \otimes_{\mathcal{O}_F} R)$ is injective; but the latter embeds in $M \otimes_{A \otimes \mathcal{O}_F} B\langle A \rangle$ which agrees with $N \otimes_{A \otimes \mathcal{O}_F} B\langle A \rangle$. This shows the exactness of the bottom line.

We denote by $\text{Snk}(R; B)$ the long exact sequence obtained from the snake lemma applied to (4.1). Note that this is also the long exact sequence obtained by applying the functor $\text{Hom}_{A \otimes \mathcal{O}_F[\tau]}(M(E), -)$ to the short exact sequence (4.2). In particular, applied with $R = B = F_\infty^s := F \otimes_K K_\infty^s$, we get

$$\text{Snk}(F_\infty^s; F_\infty^s) : 0 \longrightarrow \underline{M}_B \longrightarrow \text{Hom}_{A \otimes \mathcal{O}_F} \left(M(E), \frac{F_\infty^s \langle A \rangle}{A \otimes F_\infty^s} \right) \xrightarrow{\delta} \text{Ext}_{\text{AMot}_{F_\infty^s}}^{1, \text{reg}}(\mathbb{1}, \underline{M}_{F_\infty^s}) \longrightarrow 0.$$

Since F_∞^s is not itself a Banach algebra (but rather an ind-Banach algebra), the notation $F_\infty^s \langle A \rangle$ is understood as the direct limit

$$F_\infty^s \langle A \rangle := \lim_{L/K_\infty \text{ finite sep.}} (F \otimes_K L) \langle A \rangle$$

(the sequence (4.2) with $R = B = F_\infty^s := F \otimes_K K_\infty^s$ is exact, as a direct limit of exact sequences remains exact). By Theorem 3.1 — applied to $B = F \otimes_K L$ and then taking the direct limit over finite separable extensions L of K_∞ — we deduce that $\text{Snk}(F_\infty^s; F_\infty^s) \otimes_A \Omega_{A/\mathbb{F}}^1$ identifies with the exponential short exact sequence of E over F_∞^s .

On the other hand, the sequence $\text{Snk}(F_\infty; F_\infty)$ has the following shape (cf. [Gaz24, Proposition 4.23] and [Gaz22, Theorem 3.16]):

$$0 \rightarrow \underline{M}_B^{G_\infty} \rightarrow \text{Hom}_{A \otimes \mathcal{O}_F} \left(M(E), \frac{F_\infty \langle A \rangle}{A \otimes F_\infty} \right) \xrightarrow{\delta} \text{Ext}_{\text{AMot}_{F_\infty}}^{1, \text{reg}}(\mathbb{1}, \underline{M}_{F_\infty}) \rightarrow H^1(G_\infty, \underline{M}_B) \rightarrow 0. \quad (4.3)$$

There is no additional term on the right because $\text{id} - \tau_M$ becomes surjective³ as a map from $M \otimes_{A \otimes \mathcal{O}_F} \frac{B\langle A \rangle}{A \otimes B}$ to

$$N \otimes_{A \otimes \mathcal{O}_F} \frac{B\langle A \rangle}{A \otimes B}$$

³Indeed, $N_{F_\infty} = N \otimes_{\mathcal{O}_F} F_\infty$ is dense in $M\langle A \rangle := M \otimes_{A \otimes \mathcal{O}_F} F_\infty \langle A \rangle = N \otimes_{A \otimes \mathcal{O}_F} F_\infty \langle A \rangle$. In particular, given $y \in M\langle A \rangle$, we may write y as $n + \varepsilon$ where $n \in N_{F_\infty}$ and $\varepsilon \in M\langle A \rangle$ small enough for which the series $x := \sum_{n \geq 0} \tau_M^n(\tau^{n*}\varepsilon)$ converges in $M\langle A \rangle$. Then x is a preimage of y by $\text{id} - \tau_M$.

for these choices of R and B .

Since the functor of G_∞ -invariants is left-exact, the left-part of (4.3) agrees with the left exact sequence obtained after applying G_∞ -invariants to $\text{Snk}(F_\infty^s; F_\infty^s)$. In particular, $\text{Snk}(F_\infty, F_\infty) \otimes_A \Omega_{A/\mathbb{F}}^1$ identifies with the long exact sequence obtained by taking G_∞ -invariants of the exponential short exact sequence, and the map $\text{Ext}_{\text{AMot}_{F_\infty}}^{1, \text{reg}}(\mathbb{1}, \underline{M}_{F_\infty}) \rightarrow H^1(G_\infty, \underline{M}_B)$ identifies, up to a twist of $\Omega_{A/\mathbb{F}}^1$, with the connecting homomorphism (1.3) obtained from the exponential short exact sequence. Finally, by [Gaz22, Proposition 4.5], $\text{Snk}(\mathcal{O}_F; F_\infty)$ has the form

$$0 \rightarrow \underline{M}_B^G \rightarrow \text{Hom}_{A \otimes \mathcal{O}_F} \left(M(E), \frac{F_\infty \langle A \rangle}{A \otimes \mathcal{O}_F} \right) \rightarrow \text{Ext}_{\text{AMot}_{\mathcal{O}_F}}^{1, \text{reg}}(\mathbb{1}, \underline{M}) \xrightarrow{r_B} H^1(G_\infty, \underline{M}_B) \rightarrow \text{Cl}(\underline{M}) \rightarrow 0.$$

We deduce from the functoriality of Snk that $r_B \otimes_A \text{id}_\Omega$ identifies with t_{exp} . This proves the theorem. $\square$

Proof of Corollary 1.3. Let g be the genus of F . For a place v of F above ∞ , we let e_v and f_v denote its ramification and inertia indices respectively. Recall that F_∞ is a finite product of complete discretely valued fields and hence carries a norm $|\cdot|$ given by the sup-norm on each factor. For r a positive real number, we let $B_r \subset F_\infty$ denote the ball of radius r . The Riemann-Roch theorem implies that if $c \in \mathbb{R}$ is a real number such that

$$\sum_{v|\infty} (ce_v - 1)f_v > 2g - 2 \quad (4.4)$$

then $F_\infty = \mathcal{O}_F + B_{q^c}$. In particular, if n is such that (4.4) holds with $c = \frac{qn}{q-1}$, then the same argument as in the proof of [GM22, Proposition 2.2] shows that the $\mathbb{F}[t]$ -module

$$\text{Ext}_{\mathcal{O}_{F,\infty}}^{1, \text{reg}}(\mathbb{1}, \underline{A}(n))$$

is generated by the polylog classes $[\alpha]_n$ with $\alpha \in \mathcal{O}_F$ such that $|\alpha| < q^{\frac{nq}{q-1}}$. We may conclude from Remark 2.10. $\square$

References

- [And86] G. W. Anderson, t -motives, *Duke Math. J.* **53** (1986), no. 2, 457–502. [doi:10.1215/S0012-7094-86-05328-7](https://doi.org/10.1215/S0012-7094-86-05328-7).
- [BVK16] L. Barbieri-Viale and B. Kahn, On the derived category of 1-motives, *Astérisque*, no. 381, Société Mathématique de France, Paris, 2016, xi+254 pp. [doi:10.24033/ast.998](https://doi.org/10.24033/ast.998).
- [BH07] G. Böckle and U. Hartl, Uniformizable families of t -motives, *Trans. Amer. Math. Soc.* **359** (2007), no. 8, 3933–3972. [doi:10.1090/S0002-9947-07-04136-0](https://doi.org/10.1090/S0002-9947-07-04136-0).
- [CGL26] X. Caruso, Q. Gazda, A. Lucas, Class formulas over $A[z]$, In preparation (2026).
- [Del74] P. Deligne, Théorie de Hodge III, *Publ. Math. Inst. Hautes Études Sci.* **44** (1974), 5–77. [doi:10.1007/BF02685881](https://doi.org/10.1007/BF02685881).
- [EP14] A. El-Guindy and M. A. Papanikolas, Identities for Anderson generating functions for Drinfeld modules, *Monatsh. Math.* **173** (2014), no. 4, 471–493. [doi:10.1007/s00605-013-0543-9](https://doi.org/10.1007/s00605-013-0543-9).
- [Gaz22] Q. Gazda, Regulators in the arithmetic of function fields, unpublished preprint, [arXiv:2207.03461](https://arxiv.org/abs/2207.03461) (2022; version 3, 2026).
- [Gaz24] Q. Gazda, On the integral part of A-motivic cohomology, *Compos. Math.* **160** (2024), no. 8, 1715–1783. [doi:10.1112/S0010437X24007218](https://doi.org/10.1112/S0010437X24007218).
- [GM21] Q. Gazda and A. Maurischat, Special functions and Gauss–Thakur sums in higher rank and dimension, *J. Reine Angew. Math.* **773** (2021), 231–261. [doi:10.1515/crelle-2020-0038](https://doi.org/10.1515/crelle-2020-0038).
- [GM22] Q. Gazda and A. Maurischat, Carlitz twists: their motivic cohomology, regulators, zeta values and polylogarithms, unpublished preprint, [arXiv:2212.02972](https://arxiv.org/abs/2212.02972) (2022; version 2, 2023).
- [GM24] Q. Gazda and A. Maurischat, Residue of special functions of Anderson A-modules at the characteristic graph, *J. Number Theory* **260** (2024), 1–28. [doi:10.1016/j.jnt.2024.01.013](https://doi.org/10.1016/j.jnt.2024.01.013).

- [GM25] Q. Gazda and A. Maurischat, Pairing Anderson motives via formal residues in the Frobenius endomorphism, unpublished preprint, [arXiv:2504.01926](#) (2025; version 2, 2025).
- [GKK25] F. Głoch, D. E. Kedzierski, and P. Krasoń, Algorithms for determination of t -module structures on some extension groups, *Int. J. Number Theory* **21** (2025), no. 8, 1889–1922. [doi:10.1142/S1793042125500927](#). Corrigendum, published online 4 June 2026: [doi:10.1142/S1793042126920028](#).
- [Har19] U. Hartl, Isogenies of abelian Anderson A-modules and A-motives, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **19** (2019), no. 4, 1429–1470. [doi:10.2422/2036-2145.201612_003](#).
- [KK26] D. E. Kedzierski and P. Krasoń, Weil–Barsotti formula for t -modules, *J. Number Theory* **278** (2026), 1–25. [doi:10.1016/j.jnt.2025.04.013](#).
- [KP24] D. E. Kedzierski and P. Krasoń, On Ext^1 for Drinfeld modules, *J. Number Theory* **256** (2024), 97–135. [doi:10.1016/j.jnt.2023.09.007](#).
- [Mor18] M. Mornev, Shtuka cohomology and special values of Goss L-functions, Ph.D. thesis, Universiteit Leiden, 2018. [hdl:1887/61145](#); updated version: [arXiv:1808.00839](#).
- [PR03] M. A. Papanikolas and N. Ramachandran, A Weil–Barsotti formula for Drinfeld modules, *J. Number Theory* **98** (2003), no. 2, 407–431. [doi:10.1016/S0022-314X\(02\)00047-1](#).
- [Poo95] B. Poonen, Local height functions and the Mordell–Weil theorem for Drinfeld modules, *Compos. Math.* **97** (1995), no. 3, 349–368. [NUMDAM: CM_1995__97_3_349_0](#).
- [Sin97] S. K. Sinha, Periods of t -motives and transcendence, *Duke Math. J.* **88** (1997), no. 3, 465–535. [doi:10.1215/S0012-7094-97-08820-7](#).
- [Tae20] L. Taelman, 1 - t -motifs, in *t -Motives: Hodge Structures, Transcendence, and Other Motivic Aspects* (G. Böckle, D. Goss, U. Hartl, and M. Papanikolas, eds.), EMS Ser. Congr. Rep., EMS Publishing House, Zürich, 2020, pp. 417–439. [doi:10.4171/198-1/8](#).
- [Tae10] L. Taelman, A Dirichlet unit theorem for Drinfeld modules, *Math. Ann.* **348** (2010), no. 4, 899–907. [doi:10.1007/s00208-010-0506-6](#).
- [Woo95] S. S. Woo, Extensions of Drinfeld modules of rank 2 by the Carlitz module, *Bull. Korean Math. Soc.* **32** (1995), no. 2, 251–257.